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Rhoades-Type Fixed-Point Theorems for a Pair of Nonself Mappings

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Abstract—Using suitable conditions of weak commutativity, we prove theorems for a pair of nonself mappings which generalize earlier results due to Rhoades and Assad *et al.* Some illustrative examples and applications are also discussed.

Keywords—Metrically convex metric space, Weak commutativity, Compatible mappings, R-weak commutativity, Complete metric space, Banach space, Fixed points.

1. INTRODUCTION AND PRELIMINARIES

The study of fixed-point theorems for nonself mappings in metrically convex metric spaces was initiated by Assad and Kirk [1] which proved productive as metrically convex metric spaces offer a natural setting for proving such results. In recent years, this technique has been exploited by many authors, and by now there exists considerable literature on this topic. To mention a few, we cite [1–6].

In an attempt to generalize a theorem of Assad and Kirk [1], Rhoades [5] proved the following.

THEOREM A. *Let X be a Banach space, K a nonempty closed subset of X, and $T : K \to X$ a mapping satisfying the condition*

$$d(Tx, Ty) \leq h \max \left\{ \frac{d(x,y)}{2}, d(x,Tx), d(y,Ty), \frac{[d(x,Ty) + d(y,Tx)]}{q} \right\}$$

for all x, y in K, $0 < h < 1$, $q \geq 1 + 2h$, and T has the additional property that each $x \in \partial K$, the boundary of K, $Tx \in K$, then T has a unique fixed point.

In this paper on the lines of [3,4], we adopt definitions of 'coincidentally commuting mappings' (cf. [7]) and 'R-weakly commuting mappings' (cf. [8]) to a nonself setting and use it to prove some common fixed-point theorems on closed subsets of Banach spaces which present generalizations to Theorem A of Rhoades [5]. As an application of our main result, employing the notion of a star-shaped subset, we prove a theorem for generalized nonexpansive mappings. Two examples

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for demonstrating the validity of the hypotheses and degree of generality of our results are also presented.

Before proving our results, we collect the relevant definitions and results for our future use.

DEFINITION 1.1. *(See [3].) Let K be a nonempty subset of a metric space (X,d) and $F,T : K \to X$. Then the pair (F,T) is said to be weakly commuting if for every $x,y \in K$ with $x = Fy$, and $Ty \in K$, we have*

$$d(Tx, FTy) \le d(Ty, Fy).$$

Note that for $K = X$, this definition reduces to that of Sessa [9].

DEFINITION 1.2. *(See [4].) Let K be a nonempty subset of a metric space (X,d), and $F,T : K \to X$. Then the pair (F,T) is said to be compatible if for every sequence $\{x_n\}$ in K and from the relation*

$$\lim_{n\to\infty} d(Fx_n, Tx_n) = 0 \qquad \text{and} \qquad Tx_n \in K \qquad (n \in N),$$

it follows that

$$\lim_{n\to\infty} d(Ty_n, FTx_n) = 0$$

for every sequence $\{y_n\}$ in K with $y_n = Fx_n$, $n \in N$.

Note that for $K = X$, this definition reduces to 'compatibility' for self-mappings due to Jungck [10].

Motivated from [3], we adopt definitions of 'R-weak commutativity' and 'coincidentally commuting mappings' to the nonself setting.

DEFINITION 1.3. *Let K be a nonempty subset of a metric space (X,d), $F,T : K \to X$. Then the pair (F,T) will be called pointwise R-weakly commuting on K if for every given $x,y \in K$ with $x = Fy$, and $Ty \in K$, there exists some $R > 0$ such that*

$$d(Tx, FTy) \le R\, d(Ty, Fy). \tag{1.3.1}$$

The pair (F,T) will be called R-weakly commuting on K if for each $x \in K$, (1.3.1) holds for some $R > 0$.

By setting $R = 1$ in Definition 1.3, we get the definition of weak commutativity on K due to Hadzic and Gajic [3] (also see [5]), whereas for $R = 1$ and $K = X$, the weak commutativity is due to Sessa [9]. Also by setting $K = X$, we get the definitions of pointwise R-weak commutativity and R-weak commutativity due to Pant [8]. Here it is worth noting that the pointwise R-weak commutativity is more general than compatibility.

DEFINITION 1.4. *A pair of nonself mappings (F,T) defined on a nonempty subset K of a metric space (X,d) is said to be coincidentally commuting if $Tx, Fx \in K$ and $Tx = Fx \Rightarrow FTx = TFx$.*

Note that for $K = X$, this definition reduces to corresponding definition of Jungck and Rhoades [7] for self-mappings.

DEFINITION 1.5. *A subset K of a linear space X is said to be star-shaped if there exists at least one point $p \in K$ such that for each $x \in K$ and $t \in (0,1)$, $(1-t)p + tx \in K$. We use $\to$ to denote strong convergence and $\rightharpoonup$ to denote weak convergence.*

DEFINITION 1.6. *(See [11].) Let X be a normed linear space and K a nonempty subset of X. A mapping $T : K \to X$ is said to be demiclosed provided that if $\{x_n\} \subseteq K$, $x_n \rightharpoonup x \in K$, and $Tx_n \to y \in X$, then $Tx = y$.*

DEFINITION 1.7. *Let X be a metric space, K a nonempty subset of X, and $F, T : K \to X$. If F and T satisfy the condition*

$$d(Fx, Fy) \leq h \max \left\{ \frac{d(Tx,Ty)}{2}, d(Tx,Fx), d(Ty,Fy), \frac{[d(Tx,Fy)+d(Ty,Fx)]}{q} \right\}, \qquad (1.7.1)$$

for all x, y in K, $0 < h < 1$, $q \geq 1 + 2h$, then F is called a generalized T-contractive mapping of K into X. If we also add $h = 1$, then we call F a generalized T-nonexpansive mapping of K into X.

DEFINITION 1.8. *(See [1].) A metric space (X, d) is said to be metrically convex if for any x, y in X (with $x \neq y$), there exists a point z in X ($x \neq z \neq y$) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

LEMMA 1.9. *(See [1].) Let K be a nonempty closed subset of a metrically convex metric space X. If $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ (the boundary of K) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

2. RESULTS

We state and prove our main results as follows.

THEOREM 2.1. *Let X be a Banach space, K a nonempty closed subset of X, and $F, T : K \to X$ such that F is a generalized T-contractive mapping of K into X and*

(i) $\partial K \subseteq TK$, $FK \cap K \subset TK$,
(ii) $Tx \in \partial K \Rightarrow Fx \in K$, *and*
(iii) TK *is closed in* X.

Then there exists a coincidence point z in K. Moreover, if (F, T) is coincidentally commuting, then z remains a unique common fixed point of T and F.

PROOF. First of all, we proceed to construct the sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Let $x \in \partial K$. Then there exists a point x_0 in K such that $x = Tx_0$ as $\partial K \subset TK$. Since $Tx_0 \in \partial K$ and $Tx \in \partial K \Rightarrow Fx \in K$, we conclude that $Fx_0 \in K \cap FK \subset TK$. Let $x_1 \in K$ be such that $y_1 = Tx_1 = Fx_0 \in K$. Let $y_2 = Fx_1$. Suppose $y_2 \in K$. Then $y_2 \in K \cap FK \subseteq TK$, which implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Suppose $y_2 \notin K$. Then there exists a point $p \in \partial K$ such that

$$d(Tx_1, p) + d(p, y_2) = d(Tx_1, y_2).$$

Since $p \in \partial K \subseteq TK$, there exists a point $x_2 \in K$ such that $p = Tx_2$ so that the above equation takes the form

$$d(Tx_1, Tx_2) + d(Tx_2, y_2) = d(Tx_1, y_2).$$

Let us put $y_3 = Fx_2$. Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(a) $y_{n+1} = Fx_n$,
(b) $y_n \in K \Rightarrow y_n = Tx_n$, and
(c) $y_n \notin K \Rightarrow Tx_n \in \partial K$ and

$$d(Tx_{n-1}, Tx_n) + d(Tx_n, y_n) = d(Tx_{n-1}, y_n).$$

We denote

$$P = \{Tx_i \in \{Tx_n\} : Tx_i = y_i\},$$
$$Q = \{Tx_i \in \{Tx_n\} : Tx_i \neq y_i\}.$$

Obviously, two consecutive terms cannot lie in Q. Now, we distinguish three cases.

CASE 1. If $Tx_n, Tx_{n+1} \in P$, then

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(y_n, y_{n+1}) = d(Fx_{n-1}, Fx_n) \\ &\leq h \max\left\{\frac{d(Tx_{n-1}, Tx_n)}{2}, d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}), \right. \\ &\qquad \left. \frac{[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)]}{q}\right\} \\ &\leq hd(Tx_{n-1}, Tx_n). \end{aligned}$$

CASE 2. If $Tx_n \in P$ and $Tx_{n+1} \in Q$, then

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, y_{n+1}) \\ &= d(Tx_n, y_{n+1}) = d(Fx_{n-1}, Fx_n) \\ &\leq hd(Tx_{n-1}, Tx_n) \end{aligned}$$

in view of Case 1.

CASE 3. If $Tx_n \in Q$ and $Tx_{n+1} \in P$, then $Tx_{n-1} \in P$. Since Tx_n is a convex linear combination of Tx_{n-1} and y_n, it follows that

$$d(Tx_n, Tx_{n+1}) \leq \max\{d(Tx_{n-1}, Tx_{n+1}), d(y_n, Tx_{n+1})\}.$$

Now, if $d(Tx_{n-1}, Tx_{n+1}) \leq d(y_n, Tx_{n+1})$, then

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq d(y_n, Tx_{n+1}) = d(Fx_{n-1}, Fx_n) \\ &\leq h \max\left\{\frac{d(Tx_{n-1}, Tx_n)}{2}, d(Tx_{n-1}, y_n), d(Tx_n, Tx_{n+1}), \right. \\ &\qquad \left. \frac{[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, y_n)]}{q}\right\}. \end{aligned}$$

Now by noting that

$$\begin{aligned} d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, y_n) &\leq d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_n, y_n) \\ &\leq d(Tx_{n-1}, y_n) + d(Tx_n, Tx_{n+1}), \end{aligned}$$

one can conclude that

$$d(Tx_n, Tx_{n+1}) \leq hd(Tx_{n-1}, y_n) \leq h^2 d(Tx_{n-2}, Tx_{n-1})$$

in view of Case 2.

Otherwise, if $d(y_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_{n+1})$, then

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq d(Tx_{n-1}, Tx_{n+1}) = d(Fx_{n-2}, Fx_n) \\ &\leq h \max\left\{\frac{d(Tx_{n-2}, Tx_n)}{2}, d(Tx_{n-2}, Tx_{n-1}), d(Tx_n, Tx_{n+1}), \right. \\ &\qquad \left. \frac{[d(Tx_{n-2}, Tx_{n+1}) + d(Tx_n, Tx_{n-1})]}{q}\right\}, \end{aligned} \tag{2.1.1}$$

in view of the fact

$$\frac{d(Tx_{n-2},Tx_n)}{2} \le \frac{[d(Tx_{n-2},Tx_{n-1})+d(Tx_{n-1},Tx_n)]}{2}$$
$$\le \max\{d(Tx_{n-2},Tx_{n-1}),d(Tx_{n-1},Tx_n)\}.$$

If the maximum of the right-hand side of (2.1.1) is $[d(Tx_{n-2},Tx_{n+1})+d(Tx_n,Tx_{n-1})]/q$, then using the fact that $1+h \le q-h$ and

$$d(Tx_{n-1},Tx_n) \le d(Tx_{n-1},y_n) \le h\,d(Tx_{n-2},Tx_{n-1}),$$

we can write

$$d(Tx_{n-1},Tx_{n+1}) \le h\left[d(Tx_{n-2},Tx_{n-1})+d(Tx_{n-1},Tx_{n+1})+\frac{d(Tx_n,Tx_{n-1})}{q}\right],$$

which reduces to

$$d(Tx_{n-1},Tx_{n+1}) \le h\left\{\frac{(1+h)}{(q-h)}\right\}d(Tx_{n-2},Tx_{n-1})$$
$$\le h\,d(Tx_{n-2},Tx_{n-1}).$$

Thus, in all cases

$$d(Tx_n,Tx_{n+1}) \le h\max\{d(Tx_{n-2},Tx_{n-1}),d(Tx_{n-1},Tx_n)\}.$$

Now following the procedure of Assad and Kirk [1], it can easily be shown by induction that for $n>1$,

$$d(Tx_n,Tx_{n+1}) \le h^{n/2}\delta,$$

where

$$\delta = h^{-1/2}\max\{d(Tx_0,Tx_1),d(Tx_1,Tx_2)\}.$$

Thus, for $m,n>N$,

$$d(Tx_m,Tx_n) \le \sum_{i=N}^{\infty} d(Tx_i,Tx_{i+1}) \le \delta\sum_{i=N}^{\infty} h^{i/2},$$

which shows that $\{Tx_n\}$ is a Cauchy sequence.

First suppose that there exists a subsequence $\{Tx_{n_K}\}$ which is contained in P and TK a closed subspace of X. Since $\{Tx_{n_k}\}$ is Cauchy in TK, it converges to a point $u \in TK$. Let $v \in T^{-1}u$. Then $u = Tv$. Here one also needs to note that $\{Fx_{n_k-1}\}$ also converges to u. Using (1.7.1), one can write

$$d(Fv,Fx_{n_k-1}) \le h\max\left\{\frac{d(Tv,Tx_{n_k-1})}{2}, d(Tv,Fv), d(Tx_{n_k-1},Fx_{n_k-1}), \frac{[d(Tv,Fx_{n_k-1})+d(Tx_{n_k-1},Fv)]}{q}\right\},$$

which on making $k\to\infty$ reduces to

$$d(Fv,Tv) \le h\max\left\{0, d(Tv,Fv), 0, \frac{d(Tv,Fv)}{q}\right\},$$

yielding thereby $Fv = Tv$ which shows that v is a point of coincidence for F and T.

Since the pair (F,T) is coincidentally commuting, therefore,

$$u = Tv = Fv \Rightarrow Fu = FTv = TFv = Tu.$$

To prove that u is a fixed point of F, let on the contrary $Fu \ne u$. Then

$$d(Fu,u) = d(Fu,Fv)$$
$$\le h\max\left\{\frac{d(Fu,u)}{2}, 0, 0, \frac{[d(Fu,u)+d(Fu,u)]}{q}\right\},$$

which shows that u is a common fixed point of F and T.

The uniqueness of common fixed point follows easily. This completes the proof.

Since on the points of coincidence 'weak commutativity' implies commutativity, therefore, we can state the following corollary.

COROLLARY 2.1. *Theorem 2.1 remains true if 'coincidentally commuting property' is replaced by weakly commuting property.*

REMARK 1. Theorem 2.1 remains true if closedness of TK is substituted by the closedness of FK. Keeping in view the deduction of Theorem A from Theorem 2.1, the closedness of FK is not mentioned in the hypotheses. Note that for $T = I_k$, Theorem 2.1 reduces to Theorem A due to Rhoades [5].

THEOREM 2.2. *Let (X, d) be a complete metrically convex metric space, K a nonempty closed subset of X, $F, T : K \to X$ such that F is generalized T-contractive of K into X satisfying (i) and (ii). Suppose that*

(iv) *(F, T) is a pointwise R-weakly commuting pair, and*
(v) *maps F and T are continuous on K.*

Then F and T have a unique common fixed point.

PROOF. Proceeding as in the proof of Theorem 2.1, we suppose that there exists a subsequence $\{Tx_{n_k}\}$ which is contained in P. Further, subsequence $\{Tx_{n_k}\}$ converges to $z \in K$ as K is a closed subset of the complete metric space (X, d). Since $Tx_{n_k} = Fx_{n_k-1}$ and $Tx_{n_k-1} \in K$, the pointwise R-weak commutativity of (F, T) implies

$$d(FTx_{n_k}, TFx_{n_k-1}) \leq R\, d(Tx_{n_k}, Fx_{n_k-1}) \tag{2.2.1}$$

for some $R > 0$. Also,

$$d(FTx_{n_k}, Tz) \leq d(FTx_{n_k}, TFx_{n_k-1}) + d(TFx_{n_k-1}, Tz)\,. \tag{2.2.2}$$

Making $k \to \infty$ in (2.2.1) and (2.2.2) and using continuity of F and T, we get $d(Tz, Fz) \leq 0$ yielding thereby $Tz = Fz$.

If we assume that there exists a subsequence $\{Tx_{n_k}\}$ which is contained in Q, then analogous arguments establish the earlier conclusions.

The rest of the proof is identical to that of Theorem 2.1 after noting that at coincidence points the notions of pointwise R-weak commutativity and coincidentally commuting property are equivalent, and hence it is omitted.

Since 'pointwise R-weak commutativity' is more general then 'compatibility', therefore, we have the following corollary.

COROLLARY 2.2. *Theorem 2.2 remains true if 'pointwise R-weak commutativity' is replaced by 'compatibility'.*

REMARK 2. A comparison of Theorems 2.1 and 2.2 suggests the exploration of the possibility of some improvement in the continuity requirements of Theorem 2.2. Note that for $T = I_k$, one cannot deduce Theorem A due to additional requirement of continuity of F. This also suggests the superiority of Theorem 2.1 over Theorem 2.2.

3. AN APPLICATION

As an application of Theorem 2.1, we prove the following theorem.

THEOREM 3.1. *Let K be nonempty weakly compact star-shaped subset of a Banach space X and F a generalized T-nonexpansive mapping of K into X such that conditions (i)–(iii) (of Theorem 2.1) are satisfied. If $(I - F)$ is demiclosed and T is continuous, then F and T have a common fixed point z in K provided the pair (F, T) is coincidentally commuting.*

PROOF. Let us choose $p \in K$ such that $(1 - t)p + tx \in K$ for all $x \in K$ and all $t \in (0, 1)$. Let us put $k_n = 1 - 1/n$ $(n = 2, 3, 4, \dots)$ and define $F_n : K \to X$ by $F_n x = (1 - k_n)p + k_n Fx$ for all $x \in K$. It is easy to verify that F_n is a generalized T-contractive mapping of K into X

and F_n satisfies conditions (i)–(iii) of Theorem 2.1. Since weak topology is Hausdorff and K is weakly compact, we can conclude that K is weakly closed and therefore strongly closed. Thus, by Theorem 2.1, for each $n \geq 2$, F_n and T have a unique common fixed point, say $z_n \in K$. Now, it follows that $\{z_n\}$ has a weakly convergent subsequence and one can assume that $\{z_n\}$ itself converges to $z \in K$ weakly.

Since weakly convergent sequences are norm bounded, we conclude that $\{z_n\}$ is bounded, which amounts to say that one can find a constant $M > 0$ such that $\|z_n\| < M$ for all $n \geq 2$.

Thus, for each $n \geq 2$, we have

$$\begin{aligned}(I-F)z_n &= z_n - k_n^{-1}[F_n z_n - (1-k_n)p] \\ &= \left(1 - k_n^{-1}\right) z_n + \left(k_n^{-1} - 1\right) p,\end{aligned}$$

and hence,

$$\|(I-F)z_n\| \leq \left|k_n^{-1} - 1\right| (M + \|p\|).$$

Since $k_n^{-1} \to 1$ as $n \to \infty$, we can have $(I-F)z \to 0 \in K$. Also if $z_n \to z \in K$ and $(I-F)$ is demiclosed, it follows that $(I-F)z = 0$ giving thereby $Fz = z$. Since for each $n \geq 2$, $Tz_n = z_n$ and T is continuous, taking the limit as $n \to \infty$, one obtains $Tz = z$. Thus, we have shown that $z = Tz = Fz$. This completes the proof.

4. ILLUSTRATIVE EXAMPLES

In what follows, we furnish examples demonstrating the validity of the hypotheses and degree of generality of our results over Theorem A of Rhoades [5]. The first of these examples establishes the genuineness of Theorem 2.1 over Theorem A.

EXAMPLE 4.1. Consider $X = [1, \infty)$ with Euclidean metric d and $K = [1,3]$. Define $F, T : K \to X$ as

$$Fx = \begin{cases} x^2, & 1 \leq x \leq 2, \\ 2, & 2 < x \leq 3, \end{cases} \qquad \text{and} \qquad Tx = \begin{cases} 2x^4 - 1, & 1 \leq x \leq 2, \\ 7, & 2 < x \leq 3. \end{cases}$$

Clearly $TK = [1,31]$ and $\partial K = \{1,3\} \subset [1,31] = TK$. Further, $FK = [1,4] \Rightarrow K \cap FK = [1,3] \subset [1,31] = TK$ and $T1 = 1 \in \partial K \Rightarrow F1 = 1 \in K$, whereas $T3 = 7 \notin K$.

Note that the maps F and T are not continuous at $x = 2$, whereas the pair (F,T) is coincidentally commuting as $FT1 = 1 = TF1$.

Moreover, for $x, y \in (2,3]$, one can have $d(Fx, Fy) = 0 = h\,\frac{d(Tx,Ty)}{2}$, whereas for $x \in [1,2]$ and $y \in (2,3]$, one can write

$$\begin{aligned}d(Fx,Fy) = \left|x^2 - 2\right| &= \frac{\left|x^2-2\right|\left|x^2+2\right|}{\left|x^2+2\right|} = \frac{\left(2\left|x^4-4\right|\right)/2}{\left|x^2+2\right|} \\ &= \left(\frac{1}{x^2+2}\right)\left(\frac{d(Tx,Ty)}{2}\right).\end{aligned}$$

Finally, for $x, y \in [1,2]$,

$$\begin{aligned}d(Fx,Fy) = \left|x^2 - y^2\right| &= \frac{\left|x^2-y^2\right|\left|x^2+y^2\right|}{\left|x^2+y^2\right|} = \frac{\left(2\left|x^4-y^4\right|\right)/2}{\left|x^2+y^2\right|} \\ &= \left(\frac{1}{(x^2+y^2)}\right)\left(\frac{d(Tx,Ty)}{2}\right).\end{aligned}$$

Therefore, condition (1.7.1) is satisfied if we choose $h = \max\{1/(x^2+2), 1/(x^2+y^2)\}$. Also TK and FK are closed in X. Thus, all the conditions of Theorem 2.1 are satisfied and 1 is the unique common fixed point of F and T.

However, Theorem A of Rhoades [5] cannot be used in the context of mapping F. Otherwise for $x, y \in [1,2]$, one gets

$$d(Fx, Fy) = |x^2 - y^2| = (2|x+y|)\frac{|x-y|}{2} > h\left(\frac{d(x,y)}{2}\right)$$

because $2|x+y| > 4$, which is indeed a contradiction. Note that $1, 3 \in \partial K \Rightarrow F1 = 1 \in K$ and $F3 = 2 \in K$.

Here it is also interesting to note that the pair (F, T) is not a weakly commuting pair (cf. [3]). Otherwise, for $x = 2^{1/4}$,

$$d(FTx, TFx) = 5 > 3 - 2^{1/2} = d(Tx, Fx).$$

Our next example is constructed to demonstrate the fact that the requirement of coincidentally commuting property of the pair (F, T) is necessary in Theorem 2.1.

EXAMPLE 4.2. Consider the set of reals R equipped with Euclidean metric and $K = \{0\} \cup \{1/4^n\}_{n=1}^{\infty} \cup [1/4, 1]$. Define $F, T : K \to R$ as

$$F(0) = \frac{1}{4^2}, \quad F\left(\frac{1}{4^n}\right) = \frac{1}{4^{n+2}}, \qquad n = 0,1,2\ldots, \qquad Fx = \begin{cases} 0, & x \in \left(\frac{1}{4}, 1\right) - \left\{\frac{1}{2}\right\}, \\ \frac{1}{4^2}, & x = \frac{1}{2}, \end{cases}$$

$$T(0) = \frac{1}{4}, \quad T\left(\frac{1}{4^n}\right) = \frac{1}{4^{n+1}}, \qquad n = 0,1,2\ldots, \qquad Tx = \begin{cases} 0, & x \in \left(\frac{1}{4}, 1\right) - \left\{\frac{1}{2}\right\}, \\ \frac{1}{4}, & x = \frac{1}{2}. \end{cases}$$

It is easy to note that both the maps F and T are not continuous at the origin. Furthermore,

$$\begin{aligned} d\left(F0, F\frac{1}{2}\right) &= \left|\frac{1}{4^2} - \frac{1}{4^2}\right| = 0 \le \frac{3}{4}\left(\frac{d(T0, T1/2)}{2}\right), \\ d(F0, F1) &= \left|\frac{1}{4^2} - \frac{1}{4^2}\right| = 0 \le \frac{3}{4}.0, \\ d\left(F0, F\frac{1}{4}\right) &= \left|\frac{1}{4^2} - \frac{1}{4^3}\right| = \frac{3}{4^3} \le \frac{3}{4}\left(\frac{d(T0, T1/4)}{2}\right), \\ d\left(F0, F\frac{1}{4^2}\right) &= \left|\frac{1}{4^2} - \frac{1}{4^4}\right| = \frac{15}{16} \le \frac{3}{4}\left(\frac{d(T0, T1/4^2)}{2}\right), \end{aligned}$$

and so on. In general, $x = 1/4^n$ and $y = 1/4^m$ $(n, m = 0,1,2,\ldots)$, we have

$$\begin{aligned} d\left(F\frac{1}{4^n}, F\frac{1}{4^m}\right) &= \left|\frac{1}{4^{n+2}} - \frac{1}{4^{m+2}}\right| = \frac{1}{4}\left(\left|\frac{1}{4^{n+1}} - \frac{1}{4^{m+1}}\right|\right) \\ &\le \frac{3}{4}\left(\frac{d(T1/4^n, T1/4^m)}{2}\right). \end{aligned}$$

Finally, for $x, y \in (1/4, 1) - \{1/2\}$, one can write

$$d(Fx, Fy) = 0 \le \frac{3}{4}.0 = \frac{3}{4}\left(\frac{d(Tx, Ty)}{2}\right)$$

and for $x \in (0,1) - \{1/2\}$, and $y = 1/2$, one can have

$$d\left(Fx, F\frac{1}{2}\right) = \left|0 - \frac{1}{4^2}\right| \le \frac{3}{4}\left(\frac{d\left(|Tx, T\frac{1}{2}|\right)}{2}\right) = \frac{3}{4}\left(\frac{|0 - 1/4|}{2}\right).$$

If $x \in (1/4,1) - \{1/2\}$, and $y = 1/4^n$, then

$$d(Fx,Fy) = \left|0 - \frac{1}{4^{n+2}}\right| \le \frac{3}{4}\left(\frac{|0 - 1/4^{n+1}|}{2}\right)$$

It is easy to see that $\partial K = \{0,1\} \cup \{1/4^n\}_{n=1}^{\infty} = TK$, and $T(\partial K) = \{1/4^n\}_{n=1}^{\infty}$. Also $FK \cap K = \{0\} \cup \{1/4^n\}_{n=2}^{\infty} \subset TK$. Note that

$$T\left(\left(\frac{1}{4},1\right) - \{1/2\}\right) = 0 \in \partial K \Rightarrow F\left(\frac{1}{4},1\right) - \{1/2\} = 0 \in K,$$
$$T(0) = \frac{1}{4} \in \partial K \Rightarrow F\left(\frac{1}{4}\right) = \frac{1}{4^3} \in K,$$
$$T\left(\frac{1}{2}\right) = \frac{1}{4} \in \partial K \Rightarrow F(1) = \frac{1}{4^2} \in K,$$

and for all n, $T(1/4^n) = 1/4^{n+1} \in \partial K \Rightarrow F(1/4^{n+1}) = 1/4^{n+3} \in K$.

Clearly FK and TK are closed in X. Thus, all the conditions of Theorem 2.1 are satisfied except the coincidentally commuting property of (F,T) because for all $x \in (1/4,1) - \{1/2\}$, $Tx = Fx \Rightarrow FTx = F0 = 1/4^2 \neq 1/4 = T0 = TFx$.

Note that F and T have no common fixed point.

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